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Euclidean Markov fields of higher integer spin II. Massless case

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Abstract. The Euclidean spin-1 massless vector field and the Euclidean spin-2 massless tensor field in general covariant gauges are shown to be Markovian. However, the reflection property is not satisfied. This indicates a close connection between the requirement of indefinite metric Hilbert space in Wightman theory and the violation of the reflection property in the Euclidean theory of electromagnetic and 'gravitational' potentials.

1. Introduction

The Euclidean massless scalar field can be obtained from Nelson's theory of the massive scalar boson field (Nelson 1973a, b) by letting $m = 0$, provided the space–time dimension $n \geqslant 3$. For the spin-1 case, Gross (1975) has constructed the Euclidean electromagnetic vector potential in Lorentz gauge; however he has not shown the Markov property of the field. Lim (1975) and Yao (1976) have respectively proved the Markov property for the Euclidean electromagnetic potential in general covariant gauges and the electromagnetic field. In this paper we shall construct a Euclidean massless spin-1 vector field and a spin-2 tensor field in various covariant gauges. Both these fields are Markovian, but they do not satisfy the reflection property and hence do not lead to a Wightman theory in the Minkowski region.

Unlike the massive case, the quantization procedure of Takahashi and Umezawa (Umezawa and Takahashi 1953, Takahashi 1969) does not work for the massless case. The reason is that the Klein–Gordon divisor does not exist. For example, the Maxwell equation in terms of the vector potential A^{μ} is

$$\Lambda_{\mu\nu}(\partial)A^{\nu}(x) = 0 \tag{1.1a}$$

with

$$\Lambda_{\mu\nu}(\partial) = \Box g_{\mu\nu} - \partial_{\mu}\partial_{\nu}. \tag{1.1b}$$

The inverse operator $d^{\mu\nu}(\partial)$ for $\Lambda_{\mu\nu}(\partial)$ cannot be obtained from the equation

$$d^{\mu\nu}(\partial)\Lambda_{\sigma\nu}(\partial) = \Box\delta^{\mu}_{\nu}, \tag{1.2}$$

since the determinant of the left-hand side of equation (1.2) is zero whereas the determinant of the right-hand side is simply $\Box$. This situation corresponds to Strocchi's difficulty in the electromagnetic field related to the gauge problem (Strocchi 1970). Garding and Wightman (1964) showed in free quantum electrodynamics that weak locality and Lorentz covariance of the electromagnetic potential lead inevitably to an indefinite metric Hilbert space. The analysis of Wightman and Strocchi indicated that if A^{μ} is quasi-local or local and satisfies the Maxwell equation,

$$\partial_{\mu} F^{\mu\nu}(x)=0$$

and

$$F^{\mu\nu}(x)=\partial^{\mu} A^{\nu}(x)-\partial^{\nu} A^{\mu}(x)$$

where $F^{\mu\nu}$ is the electromagnetic field tensor, then $F^{\mu\nu}$ is trivial.

To circumvent these difficulties it has been usual to follow one of the following two methods. We can either abandon the requirement that A^{μ} should be relativistic covariant and quasi-local or local; or we can accept the fact that in a covariant and local theory, Maxwell equations are not satisfied as operator equalities. The first method is known as the Coulomb (or radiation) gauge formalism, which has the gauge condition $\boldsymbol{\nabla} \cdot \boldsymbol{A}=0$ and the Hilbert space has positive definite metric. However, the theory is no longer manifestly covariant and local, so it is necessary to supply a gauge term with each Lorentz transformation in order to obtain covariance of the Coulomb condition. We shall adopt the second approach which is known as the Gupta–Bleuler formalism. This is a local and covariant theory, but the underlying Hilbert space is not positive definite. The Maxwell equations are no longer satisfied in the whole indefinite Hilbert space. The physical states $|\phi\rangle$ are defined as those satisfying the non-local subsidiary condition

$$\partial_{\mu} A^{\mu(+)}(x)|\phi\rangle=0$$

where $\partial_{\mu} A^{\mu(+)}$ is the positive-frequency part of the operator $\partial_{\mu} A^{\mu}$. Then the Hilbert space spanned by the physical states $|\phi\rangle$ has a positive metric. However, this physical subspace is not dense in the original indefinite Hilbert space, so the Maxwell equations are only satisfied in the sense that they hold when one takes the matrix elements of these equations between physical states†.

2. Free Euclidean electromagnetic potential

The free propagator for the electromagnetic potential A^{μ} in the Gupta–Bleuler formalism is

$$D^{\mu\nu}(p)=\langle\tilde{A}^{\mu}\tilde{A}^{\nu}\rangle=g^{\mu\nu}p^{-2}. \tag{2.1}$$

This propagator determines the Lorentz gauge for A^{μ}. The corresponding Euclidean propagator is given by

$$S_{ij}(p)=A_{i\mu}A_{j\nu}D^{\mu\nu}(\boldsymbol{p}, \mathrm{i}p_0)=\delta_{ij}p^{-2}, \tag{2.2}$$

where $A_{i\mu}=1$ for $i=\mu=1,2,3$; $A_{40}=\mathrm{i}$ and $A_{i\mu}=0$ otherwise. Here we have used the same notation p^2 for both the Euclidean norm $p_{\mathrm{E}}^2=-(\boldsymbol{p}^2+p_4^2)$ and the Minkowski norm $p_{\mathrm{M}}^2=p_0^2-\boldsymbol{p}^2$. S_{ij} is positive definite so we can construct a Euclidean vector potential $\mathscr{A}_i$ with $E[\tilde{\mathscr{A}}_i\tilde{\mathscr{A}}_j]=\delta_{ij}p^{-2}$. In order to consider the Euclidean electromagnetic vector potential in a wide class of covariant gauges we need the following gauge transformation:

$$\mathscr{A}_i \rightarrow \mathscr{A}_i'=\mathscr{A}_i+\partial_i V \tag{2.3}$$

† After the completion of this paper, the author was able to show that Euclidean electromagnetic potential in Coulomb (or radiation) gauge is also Markovian.

where V is a real scalar random field independent of $\mathscr{A}$, such that the propagator of the transformed field has the general form

$$S_{ij}(p)=\left(\delta_{ij}-(F(p^2)-1)\frac{p_ip_j}{p^2}\right)\frac{1}{p^2} \tag{2.4}$$

where $F\geqslant 0$ is a measurable function which parametrizes a family of covariant gauges for $\mathscr{A}_i$. We require $F\geqslant 0$ so that the Euclidean field has a positive semi-definite metric; however, we ignore the fact that the relativistic field should have a positive metric. In fact, in the Minkowski region there is a dipole ghost (except for $F=1$ corresponding to the Lorentz gauge) and the Gupta–Bleuler formalism has to be used. We note that the limit as $m\to 0$ of the Euclidean propagator for the Proca field does not exist, but the propagator (2.4) is just the massless limit of the propagator of the vector meson in R_ξ gauges in our previous paper.

Now we can define the Euclidean one-particle space as the completion of the real vector test function space $\mathscr{S}^4(\mathbb{R}^4)$ with respect to the norm given by the inner product

$$\langle f,g\rangle_{\mathscr{H}}=\sum_{i,j} f_i(x)S_{ij}(x-y)g_j(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.5}$$

The Euclidean vector potential with a family of covariant gauges parametrized by the gauge function F is defined as the generalized Gaussian random vector field over $\mathscr{H}$ with mean zero and covariance given by

$$E[\mathscr{A}(f)\mathscr{A}(g)]=\langle f,g\rangle_{\mathscr{H}}.$$

The Euclidean propagator has an inverse of the form

$$S_{ij}^{-1}(p)=p^2\delta_{ij}+(F^{-1}(p^2)-1)p_ip_j. \tag{2.6}$$

Definition 1. The covariant gauge function $F(p^2)$ characterizing the Euclidean electromagnetic vector potential $\mathscr{A}$ indexed by $\mathscr{H}$ is called a Markov gauge function if $F\geqslant 0$ and its inverse F^{-1} exists (except for the case $F=0$) as a polynomial in p^2.

For the case $F>0$, the existence of F^{-1} as a polynomial in p^2 implies that S_{ij}^{-1} is local, which guarantees the Markov property. However, for the case $F=0$, corresponding to Landau (or transverse) gauge, the Markov property of the vector potential in this gauge is not so obvious, because now S_{ij} is singular and cannot be inverted. Furthermore, S_{ij} is positive semi-definite, so the Euclidean one-particle space is the quotient space $\mathscr{H}/\ker\|...\|_{\mathscr{H}}$.

Theorem 1. The Euclidean electromagnetic vector potential in covariant gauges parametrized by the gauge function F is Markovian if F is a Markov gauge.

Proof. For $F>0$, S_{ij} has a local inverse given by (2.6), the proof of Markov property given by Nelson (1973a, b) for the scalar case applies.

For $F=0$ (i.e. Landau gauge) we shall restrict ourselves to test functions satisfying $\Sigma_i\,\partial_if_i(x)=0$, so that the inverse of S_{ij} exists in this subspace and Nelson's argument can be used. Let $\mathscr{O}\subset\mathbb{R}^4$ be an open set and let $\mathscr{H}_1$ be the set of distribution vector fields f_i such that

$$\sum_i \partial_if_i(x)=0 \qquad \text{and} \qquad \|f\|^2_{\mathscr{H}_1}=\sum_i\int \tilde{f}_i(p)p^{-2}\tilde{f}_i(p)\,\mathrm{d}^4p<\infty.$$

Let Σ_1 be the Borel-ring generated by the Gaussian field $\mathscr{A}_i$ over $\mathscr{K}_1$ and $\Sigma_1(\mathcal{O})$ be the σ-ring generated by $\mathscr{A}(f)$, $\sup f \subset \mathcal{O}$, $f \in \mathscr{K}_1$. Let $f \in \Sigma_1(\mathcal{O})$ and let $e_{\mathcal{O}'}$ be the projection onto $\mathcal{O}'$, the complement of $\mathcal{O}$. Then if $h \in C^\infty(\mathring{\mathcal{O}}')$ where $\mathring{\mathcal{O}}'$ is the interior of $\mathcal{O}'$, we have

$$\langle e_{\mathcal{O}'} f, h\rangle_{\mathscr{L}^2} = \sum_{i,j} \left\langle (e_{\mathcal{O}'}\tilde{f})_i, \frac{1}{p^2} p^2(\delta_{ij} - p^{-2} p_i p_j)\tilde{h}_j \right\rangle_{\mathscr{L}^2}$$

since $\Sigma_i\, p_i (e_{\mathcal{O}'}\tilde{f})_i = 0$,

$$= \sum_{i,j} \left\langle (e_{\mathcal{O}'} f)_i, \frac{1}{p^2}(p^2\delta_{ij} - p_i p_j)\tilde{h}_j \right\rangle_{\mathscr{L}^2}.$$

Now $\Sigma_i\, (p^2\delta_{ij} - p_i p_j)\tilde{h}_j \in \mathscr{K}_1(\mathring{\mathcal{O}})$, so

$$= \sum_{i,j} \langle (e_{\mathcal{O}'}\tilde{f})_i, (p^2\delta_{ij} - p_i p_j)\tilde{h}_j\rangle_{\mathscr{K}_1}$$

$$= \sum_{i,j} \langle \tilde{f}_i, (p^2\delta_{ij} - p_i p_j)\tilde{h}_j\rangle_{\mathscr{K}_1}$$

$$= \sum_{i,j} \langle \tilde{f}_i, (p^2\delta_{ij} - p_i p_j)p^{-2} h_j\rangle_{\mathscr{L}^2}$$

Since $\Sigma_i\, p_i \tilde{f}_i = 0$,

$$= \langle \tilde{f}, \tilde{h}\rangle_{\mathscr{L}^2} = 0.$$

Hence $\sup e_{\mathcal{O}'} f \subset \partial\mathcal{O}$, and the field in Landau gauge is Markovian.

What we have just proved is the Markov property for a Euclidean electromagnetic potential satisfying the Lorentz condition. We note that only in the Landau gauge does the electromagnetic potential satisfy the Lorentz condition as an operator identity. Thus we obtain the same theory as Gross (1975) where test functions are subjected to the condition $\Sigma_i\, \partial_i \mathscr{A}_i = 0$ and the covariance is $\delta_{ij} p^{-2}$. However, we remark that in no gauge do we get a Wightman theory. For example, $F = 0$ leads to a non-local theory and $F = 1$ leads to an indefinite metric. It is not surprising that the Euclidean electromagnetic potential does not satisfy the reflection property.

Theorem 2. The Euclidean electromagnetic vector potential $\mathscr{A}$ does not satisfy the reflection property.

Proof. The proof is exactly the same as for the massive vector field in R_ξ-gauges. The 4–4 component of the two-point Schwinger function contains the term $p_4^2 p^{-2}(1 - F(p^2))p^{-2}$ which allows test functions of finite $\mathscr{K}_1$ norm localized at the hyperplane $x_4 = 0$ of the form

$$f_i(x) = f_i(\boldsymbol{x}) \otimes \delta(x_4)$$

with $f_4 \neq 0$ and $f_i \in \mathscr{S}(\mathbb{R}^3)$. For such a test function we have

$$\tau(\rho) f_4(x) = -f_4(x)$$

where $\tau(\rho)$ is a representative of reflection ρ. Therefore $\tau(\rho)\mathscr{A}(f) \neq \mathscr{A}(f)$.

We might now ask for a property which, while more general than the reflection property (so as to include theories like electrodynamics and gravitation), yet still excludes very non-local theories like the one with propagator of the form $(-\Delta+m^2)^n$, $n>1$. Such a property is known as the classical Markov property which can be formulated as follows. Let $\mathcal{H}$ be the completion of $\mathcal{S}(\mathbb{R}^4)$ in the norm defined by the covariance function of the random field in question. For any open set $\mathcal{O}\subset\mathbb{R}^4$, let $\mathcal{H}(\mathcal{O})$ be the (closed) subspace generated by $\{f\in\mathcal{H}, \sup f\subset\mathcal{O}\}$ and let $\Sigma_{\mathcal{O}}$ be the σ-algebra generated by $\{\Phi(f)|f\in\mathcal{H}(\mathcal{O})\}$. Let $\mathcal{H}_0(\mathcal{O})$ be the subspace of $\mathcal{H}(\mathcal{O})$, consisting of measures, and let $\Sigma_{0\mathcal{O}}$ denote the Borel σ-ring generated by $\{\Phi(f)|f\in\mathcal{H}_0(\mathcal{O})\}$. For any subset $\mathcal{U}\subset\mathbb{R}^4$, denoted by $\Sigma_{0\mathcal{U}}$ the intersection $\bigcap\{\Sigma_{0\mathcal{O}}|\mathcal{O}\supset\mathcal{U}, \mathcal{O}\text{ open}\}$. Then we say that a field Φ satisfies the classical Markov property if, for every function $\mathcal{G}:\Omega\to\mathbb{R}$ which is $\Sigma_{\mathcal{O}}$-measurable, and every open set $\mathcal{O}\subset\mathbb{R}^4$,

$$E[\mathcal{G}|\Sigma_{\mathcal{O}'}]=E[\mathcal{G}|\Sigma_{0\partial\mathcal{O}}]$$

holds, where $\mathcal{O}'$ is the complement of $\mathcal{O}$ in $\mathbb{R}^4$, and $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$.

The generalized random fields defined by the Euclidean electromagnetic potential in various Markov gauges are such that $\Sigma_{0\partial\mathcal{O}}$ coincides with $\Sigma_{\partial\mathcal{O}}=\bigcap_{\mathcal{O}_i}\{\Sigma_{\mathcal{O}_i}|\mathcal{O}_i\subset\partial\mathcal{O}\}$. Hence the classical and Nelson's definitions of Markovicity coincide. These fields therefore also satisfy the classical Markov property.

3. Euclidean massless spin-2 tensor field

In classical theory, massless spin-2 particles can be described by a rank-two tensor $\psi^{\mu\nu}$ satisfying the wave equation

$$\Box\psi^{\mu\nu}(x)=0, \tag{3.1}$$

and the following subsidiary conditions:

$$\psi^{\mu\nu}(x)=\psi^{\nu\mu}(x) \tag{3.2}$$

$$\psi^{\mu}_{\mu}(x)=0 \tag{3.3}$$

$$\partial_\mu\psi^{\mu\nu}(x)=0. \tag{3.4}$$

Just like the case of the electromagnetic vector potential, we cannot obtain all these equations by using the variational principle. This means that unlike the massive case, the Umezawa–Takahashi method of quantization does not apply here. The analysis carried out by Bracci and Strocchi (1972, 1975) has shown that gauge problems in a massless spin-2 field give rise to difficulties analogous to those that exist in quantum electrodynamics. Their results indicate that a local and covariant description of a massless spin-2 field is possible only in a Hilbert space with indefinite metric. In other words, the Gupta–Bleuler formalism needs to be used. The presence of unphysical (or ghost) states in the theory requires the subsidiary conditions (3.3) and (3.4) to hold only on physical states $|\phi\rangle$ in the form

$$\psi^{\mu(+)}_{\mu}(x)|\phi\rangle=0 \tag{3.5}$$

$$\partial_\mu\psi^{\mu\nu(+)}(x)|\phi\rangle=0 \tag{3.6}$$

where $\psi^{\mu(+)}_{\mu}$ and $\psi^{\mu\nu(+)}$ denote the positive frequency parts of ψ^{μ}_{μ} and $\psi^{\mu\nu}$ respectively. The Fourier transform of the free propagator is then given by

$$D^{\mu\nu\rho\sigma}(p)=\langle\tilde{\psi}^{\mu\nu}\tilde{\psi}^{\rho\sigma}\rangle=\frac{1}{2p^2}(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}). \tag{3.7}$$

This gives a theory of a massless spin-2 field in a gauge analogous to the Lorentz gauge in electromagnetic potential.

The corresponding Euclidean propagator is

$$S_{ijmn}(p) = A_{i\mu}A_{j\nu}A_{m\rho}A_{n\sigma}D^{\mu\nu\rho\sigma}(\boldsymbol{p}, \mathrm{i}p_0) = \frac{1}{2p^2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \tag{3.8}$$

which is positive definite and has a local inverse. A Euclidean Markov tensor field ψ_{ij} with S_{ijmn} as covariance can be constructed in the usual way. In order to obtain a theory of a massless spin-2 Euclidean tensor field in a wide class of covariant gauges, the following gauge transformation is necessary:

$$\Psi_{ij} \to \Psi'_{ij} = \Psi_{ij} + \partial_i B_j + \partial_j B_i + \delta_{ij} C \tag{3.9}$$

where B_i and C are generalized random vector and scalar fields respectively, independent of each other, and independent of Ψ_{ij}. The Fourier transform of the Euclidean propagator of the transformed field has the general form

$$S_{ijmn}(p) = \frac{1}{2p^2}\bigg((\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} + a\delta_{ij}\delta_{mn}) + \frac{F(p^2)}{p^2}(\delta_{ij}p_m p_n + \delta_{mn}p_i p_j)$$
$$+ \frac{G(p^2)}{p^2}(\delta_{im}p_j p_n + \delta_{in}p_j p_m + \delta_{jm}p_i p_n + \delta_{jn}p_i p_m) + \frac{H(p^2)}{p^4}p_i p_j p_m p_n\bigg) \tag{3.10}$$

where a is a real constant greater than or equal to $-\frac{1}{2}$; F, G and H are positive measurable functions (polynomial in p^2). It is positive definite (or semi-definite) and has an inverse

$$S^{-1}_{ijmn}(p) = 2p^2(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + a_1\Box\delta_{ij}\delta_{mn} + a_2\delta_{ij}p_m p_n + a_3\delta_{mn}p_i p_j$$
$$+ a_4(\delta_{im}p_j p_n + \delta_{in}p_j p_m + \delta_{jm}p_i p_n + \delta_{jn}p_i p_m) \tag{3.11}$$

with

$$a_1 = -[2a + a_2(a+F)](4a+1+F)^{-1},$$
$$a_2 = [3F+G+H-(4a+1)][(1+F+G+H)(4a+1+F)-(1+F)(4F+G+H)]^{-1},$$
$$a_3 = (4aG+2FG-4F)(5+F)^{-1}, \qquad a_4 = -4G(2+G)^{-1}.$$

The Euclidean one-particle space $\mathscr{K}$ can be defined as the completion of the symmetric test function space $\mathscr{S}'^0(\mathbb{R}^4)$ with respect to the inner product

$$\langle f, g\rangle_{\mathscr{K}} = \sum_{\substack{i<j \\ m<n}} \langle \tilde{f}_{ij}, S_{ijmn}\tilde{g}_{mn}\rangle_{\mathscr{L}^2}. \tag{3.12}$$

The Euclidean massless spin-2 tensor field Ψ (or Euclidean gravitational potential in flat space) in covariant gauges characterized by B_i and C is defined as the generalized Gaussian tensor field with mean zero and covariance $E[\Psi(f)\Psi(g)] = \langle f, g\rangle_{\mathscr{K}}$. Now we have four gauge parameters as compared to one gauge parameter in the electromagnetic potential. This is expected because the gauge transformation (3.9) for the spin-2 potential involves a four-vector field and a scalar field, whereas in the spin-1 potential there is only a single scalar field. This clearly indicates that the spin-2 propagator will be of much larger variety than its spin-1 counterparts.

Definition 2. The covariant gauges characterized by the gauge functions F, G, H and parameter a are called Markov gauges if a_1, a_2, a_3 and a_4 are all local. In other words F, G, H and a define a Markov gauge if S^{-1}_{ijmn} exist as a local operator. The above definition does not include the transverse (or Landau) gauge with $a=-\frac{2}{3}$, $F=+\frac{2}{3}$, $G=-1$ and $H=+\frac{4}{3}$, and such that $\Sigma_i\, p_i S_{ijmn}(p)=0$. We shall see that in this gauge we also obtain a Markov field.

Theorem 3.
(i) The tensor field Ψ with Markov gauges satisfies the Markov property.
(ii) The tensor field Ψ with the transverse gauge satisfies the Markov property.

Proof.
(i) Proof for (i) is just as before, S^{-1}_{ijmn} exists as a local operator. Hence the Markov property follows from Nelson's argument.
(ii) For the transverse (or Landau) gauge, the Euclidean propagator is singular and can be written in the following form:

$$S_{ijmn}(p)=\frac{1}{2p^2}(d_{im}d_{jn}+d_{in}d_{jm}-\tfrac{2}{3}d_{ij}d_{mn})$$

with $d_{ij}=\delta_{ij}-p^{-2}p_ip_j$. We have $\Sigma_i\, p_iS_{ijmn}$ and $\Sigma_i\, S_{iimn}=0$. Now consider the subspace $\mathscr{K}_1$ of a test function, satisfying the following conditions: $f_{ij}=f_{ji}$, $\Sigma_i\, f_{ii}=0$ and $\Sigma_i\, p_i\tilde{f}_{ij}=0$. For $f,g\in\mathscr{K}_1$, the inner product in $\mathscr{K}$ reduces to

$$\langle f,g\rangle_{\mathscr{K}}=\left\langle \tilde{f}_{ij},\frac{\delta_{im}\delta_{jn}}{p^2}\tilde{g}_{mn}\right\rangle_{\mathscr{L}^2}.$$

S_{ijmn} maps $\mathscr{K}_1$ into $\mathscr{K}_1$ since for any $h\in\mathscr{K}_1$,

$$\sum_i p_iS_{ijmn}\tilde{h}_{mn}=0 \qquad \text{and} \qquad \sum_i S_{iimn}\tilde{h}_{mn}=0.$$

The proof depends crucially on two points. Firstly, S_{ijmn} maps any element of $C^\infty(\mathcal{O})$ to an element belonging to $K_1(\mathcal{O})$. Second point is that the inverse of $(\delta_{im}\delta_{jn})p^{-2}$ is a local operator. The rest of the proof goes exactly like theorem 2 for the electromagnetic potential in the Landau gauge.

A special case of interest is the following relativistic propagator for a graviton, considered by Isham and Abdus Salam (1973):

$$D^{\mu\nu\rho\sigma}(p)=(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-2cg^{\mu\nu}g^{\rho\sigma})\frac{1}{2p^2} \qquad (3.13)$$

where c is a real parameter. In order to get a positive semi-definite Euclidean propagator we need to get a traceless S_{ijmn}, namely

$$\begin{aligned}S_{ijmn}(p)&=(A_{i\mu}A_{j\nu}+\tfrac{1}{4}\delta_{ij}g_{\mu\nu})(A_{m\rho}A_{n\sigma}+\tfrac{1}{4}\delta_{mn}g_{\rho\sigma})D^{\mu\nu\rho\sigma}(\boldsymbol{p},\mathrm{i}p_0)\\&=\frac{1}{2p^2}(\delta_{im}\delta_{jn}+\delta_{in}\delta_{jm}-\tfrac{1}{2}\delta_{ij}\delta_{mn}).\end{aligned} \qquad (3.14)$$

Now S_{ijmn} is independent of c and is positive semi-definite. The Euclidean one-particle space is then the quotient space $\mathscr{K}/\ker\|\ldots\|_{\mathscr{K}}$. Again the Euclidean spin-2 potential in these gauges is Markovian. We need only consider the traceless symmetric subspace $\mathscr{K}_1$

of $\mathscr{K}$; then S_{ijmn} maps any $h \in C^{\infty}(\mathcal{O})$ into an element of $\mathscr{K}_1(\mathcal{O})$. Then it is not difficult to verify the Markovicity of such a field.

Theorem 4. The tensor field Ψ with covariant gauges does not satisfy the reflection property.

Proof. The proof is exactly similar to that for the electromagnetic potential; therefore we shall omit.

Thus, we have another example of a Euclidean field which is Markovian but does not satisfy the reflection property. Again it does not lead to a Wightman theory in the Minkowski region. Actually for the Euclidean electromagnetic and 'gravitational' potentials, $\tau(\rho)$ may be interpreted as CT rather than T, where C is charge conjugation and T is time reversal. Therefore we conclude that a Markov property alone is not enough to guarantee the existence of a Wightman field; the reflection property is also required to be satisfied.

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